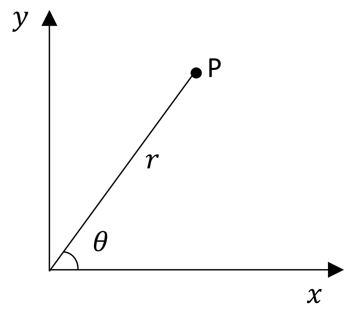
Polar Coordinates Cheat Sheet

The familiar x and y axes of the 2D plane are just one set of coordinates which can be used to describe each point in the plane. Another set which could be used are called polar coordinates where each point is described by its radial distance from the origin and an angle.

Representing a Point

Any point P can be described by the coordinates (r, θ), where r is the radial distance from the point to the origin (also referred to as the pole) and θ is the angle subtended by the radial line connecting the point to the origin and the x-axis. Note that the x axis is often called the initial line.



Trigonometry and the Pythagorean theorem can be used to relate r and θ to x and y in order to convert between coordinate systems:

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan\left(\frac{y}{x}\right)$$

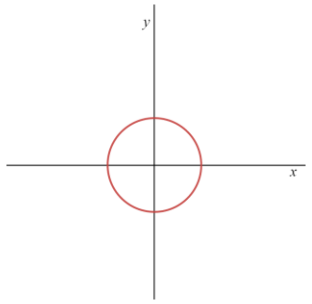
$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Example 1: Rewrite the equation $ax - by = 0$ in polar coordinates.

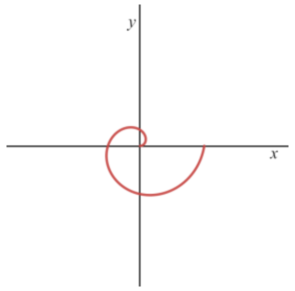
Directly substituting in the equations for x and y.	$a\sin\theta - b\cos\theta = 0$
Rearranging for a simpler form.	$\tan\theta = \frac{b}{a}$

Simple Polar Curves

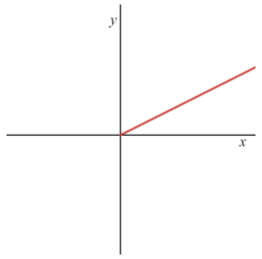
The simplest possible polar curves are shown below. These provide a good starting point for thinking about harder polar curves.



A circle of radius a is given by $r = a$



A spiral is given by $r = a\theta$



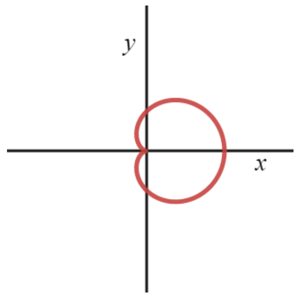
A half-line is given by $r = \theta$, the half line makes an angle θ with the x axis.

Sketching Complex Polar Curves

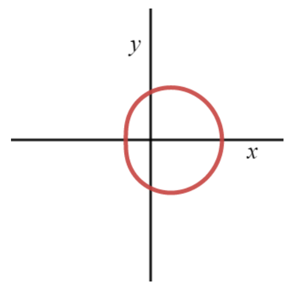
Complex polar curves can be difficult to intuit. Given a function $r = r(\theta)$, the general shape of this curve can be investigated by looking at key points such as when r is maximum or zero.

There are two common examples of complex curves worth remembering the general shape of:

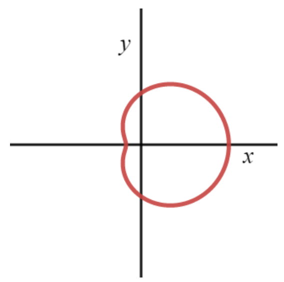
- $r = a\cos n\theta$ or $r = a\sin n\theta$. If n is odd, then n loops are produced, if n is even then $2n$ loops are produced.
- $r = a(b + \cos\theta)$. This is best analysed on a case-by-case basis:
    - If $|b| \geq 2$, then it produces an egg shape.
    - If $1 < |b| < 2$, then it produces an egg shape with a dimple on one side.
    - If $|b| = 1$, then it produces a cardioid (a heart shape curved).



$r = 1 + \cos\theta$, this is a cardioid.



$r = 2.1 + \cos\theta$, this is egg-shaped.



$r = 1.5 + \cos\theta$, this is egg-shaped with a dimple.

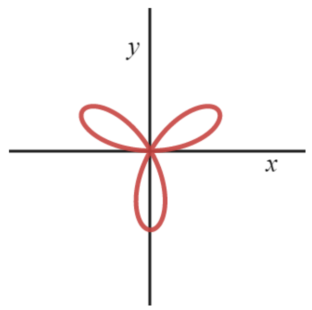
When analysing an equation of the above form but with $\sin\theta$ instead of $\cos\theta$ we can use the identity $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$ to see that it gives the same curve just rotated anti-clockwise by $\frac{\pi}{2}$. This is exactly analogous as to how $y = f(x - a)$ is just $y = f(x)$ shifted by a in the x direction.

Example 2: Sketch the curve given by $r = \sin 3\theta$.

First, by noting that this is periodic every $\frac{2\pi}{3}$, the amount of work needed can be reduced as only the key elements of the curve between $\theta = 0$ and $\theta = \frac{2\pi}{3}$ are required to sketch the full curve. These are:

θ:	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
r:	0	1	0	-1	0

These points alone produce 2 loops, by rotating the pattern round and accounting for overlap, the desired 3 loop pattern is obtained.



Area Enclosed by a Polar Curve (A Level Only)

In cartesian co-ordinates, integrating to find the area under the curve can be thought of as, splitting the curve into N rectangles of width dx and height $y = f(x)$ then summing all the areas of these rectangles in the limit of $dx \to 0, N \to \infty$. In polar coordinates, something similar can be done. The curve is split into triangles of base $rd\theta$ and height r. In the equivalent limit of $d\theta \to 0$, the area enclosed by the curve is equivalent to the sum of all the triangles' areas. Thus, the following formula for the enclosed area A, can be used:

$$A = \frac{1}{2}\int r^2 d\theta$$

A useful check of this reasoning is to see that it produces the correct formula for the area of a circle of radius a:

$$A = \frac{1}{2}\int_0^{2\pi} a^2 d\theta = \frac{1}{2}a^2\theta\Big|_0^{2\pi} = \pi a^2$$

The following identities are useful for finding areas enclosed by curves with trigonometric properties:

$$\cos 2\theta \equiv 1 - 2\sin^2\theta \equiv 2\cos^2\theta - 1$$

Example 3: Find the area A, bounded by the half lines $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{2}$ and the curves $r = 2\cos\theta$ and $r = \sin 2\theta$.

Begin by noting the area asked for is the difference between the areas enclosed by each curve and the half lines.	$A = A_{big} - A_{small}$
Proceed by finding A_{big}, the area enclosed by the curve $r = 2\cos\theta$ and the half lines.	$A_{big} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$
Using the identity re-arranged from earlier $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, the integral can be evaluated.	$A_{big} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$ $\Rightarrow A_{big} = \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big\|_{\pi/3}^{\pi/2}$ $= \frac{\pi}{6} - \frac{\sqrt{2}}{2}$
A_{small} is similarly found.	$A_{small} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi/2} \sin^2 2\theta\, d\theta$
The other identity can be used to evaluate the integral.	$A_{small} = \frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 - \cos 4\theta d\theta$ $= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big\|_{\pi/3}^{\pi/2}$ $= \frac{\pi}{24} - \frac{\sqrt{3}}{32}$
A can now be found as the difference.	$A = \frac{\pi}{8} - \frac{7\sqrt{3}}{32}$

Example 4: The curve given by $r = a + 4\cos\theta$ where $a > 0, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, encloses a total area of 100. Find the value of a.

The problem can be set up by using the formula for the enclosed area. Finding the area in terms of a will lead to the solution.	$100 = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (a + 4\cos\theta)^2 d\theta$
Proceed by expanding the brackets.	$100 = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} a^2 + 16\cos^2\theta + 8\cos\theta\, d\theta$
We use $\cos^2\theta \equiv \frac{1}{2}(1 + \cos 2\theta)$ to compute the integral.	$100 = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} a^2 + 8(1 + \cos 2\theta + \cos\theta) d\theta$ $\Rightarrow 200 = a^2\theta + 8\theta + 4\sin 2\theta + 8\sin\theta\,\|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ $200 = 2\pi(a^2 + 8) + 4\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right)$ $+ 8\left(\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}\right)$
By simplifying an equation for a is found.	$200 = 2\pi(a^2 + 8) + 4(-1 - 1) + 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$ $\Rightarrow 200 = 2\pi(a^2 + 8) - 8$ $\Rightarrow \frac{194}{2\pi} - 8 = a^2$
This has solutions $a = \pm\sqrt{\frac{194}{2\pi} - 8}$ but the question specifies $a > 0$.	$a = \sqrt{\frac{194}{2\pi} - 8}$